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Displacement Study of Cantilever Beam Using Theory of Elasticity in Plane Stress

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List of Contents:

List of figures:

List of tables:

Displacement Study of Cantilever Beam Using Theory of Elasticity in Plane Stress

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Abstract

The theory of elasticity is a part of solid mechanics with isotropic materials subject to elastic stresses, strains, and displacements. The tensile elasticity module and the elastic limit describe the elasticity behavior of a material in engineering. Plane stress that called two dimensional has many engineering applications in structural analysis, especially for plane elements. Application of the plane stress method used to analysis such as beam web frame of tanks, Pressure vessels and the thickness of such kind applications are small when compared to the length of the object and mostly the loading conditions are in plane only. This literature discusses the elasticity theory of plane stress and the relationship between strain-displacement in order to find or calculation displacement in components and compare it with its methods such as virtual work method and computer analysis for some methods such as (stiffness method, plane stress method, solid element and FEM method). When calculated displacement by theory of elasticity, it has an option that displacement not verified with elementary books on the strength of materials by shear stress cause to distortion of the cross section in the ends by boundary options. but it has another option and polynomial method (stress function) that verified with elementary books on the strength of materials and with stiffness method that it must be selected, the difference is zero. Also, in models contain effect of shear displacement was more difference by 5.54 % increased, as well as it was near to second option that contain shear effect. but other methods model as (plane stress, solid element and FEM) have a little different (2.23%, 0.46% and 3.88%) respectively, because they are more accurate 3D model methods, as well as by meshing element and support condition.

Keywords: Plane stress, Displacement, Polynomial, Elasticity, and Shear stress.

1.Introduction

The theory of elasticity is a solid mechanical branch which deals with elastic stresses, strains, and displacements of isotropic materials. There are some fundamental assumptions (the material is perfectly elastic, homogeneous and isotropic in all directions) in the elasticity principle that are used to simplify the equations. There are several concerns in elasticity theory under two conditions, plane stress, in this case, plane stress is called two-dimensional, stress is subjected to only plane faces of the element, and all stresses act parallel to the plane axis. and plane strain, in this case, compared to the others, if one dimension is very large, the main strain is constrained in the direction of the longest dimension and can be assumed as zero.

Displacement is the main component of physical analysis, and in order to solve their problems, the theory of elasticity began with an important relationship between components such as strain and displacement and stress strain. In the methodology section describes the relationship and the derived equations used to calculate displacement in both horizontal and vertical directions and polynomial method. For bending and displacement elasticity theory, there are two options and it has two equations, caused by boundary conditions, for both horizontal and vertical displacement. In order to select the best rule generated in the methods, it is important to compare it with another method in elementary mechanics that selected the virtual work method to do it in the same section and with other methods that illustrated in section result and discussion. In the first option method, the displacement by shear stress and loaded end beam of the value from the equation ($PL^3/3EI$) remains only coinciding with the value typically derived from the material strength in elementary books, which was checked with it. And this value coinciding with other method such as stiffness method, but the results solid method was near than the results of displacement elasticity equation theory. The Finite element method and plane stress have more displacement value may be cause to concentration force under load at the end of beam, meshing element and support condition. The highest displacements are induced in the sap2000 model containing the shear area effect, when compared with material strength in elementary books and elasticity equations theory. The theory of elasticity of plane stress and the relationship between strain-displacement, it is main work for this literature review in order to find displacement in components and compare it with its methods and computer analysis.

2. Review Literature

The relationships between the stress components and the strain components have been experimentally developed, known as Hooke's rule. The magnitude of the element's unit elongation is given by the law of Hooke's equation. In which the elasticity modulus in tension is elasticity. In comparison with acceptable stresses, the engineering structure materials have a very elevated module, and the unit elongation is a very small number. In which Poisson ratio is a constant, called the ratio of Poisson. For multiple materials the Poisson ratio can be estimated to be 0.25. It is structural steel for usually taken at the amount of 0.3 (Timoshenko et al., 1951). If a deformable body is forced to lengthen but also contract laterally, it is called the Poisson ratio. The deformations not only continue or contract line segments, but also cause them to change direction. When choosing two-line segments initially perpendicular to one another, the angle change is called a shear strain. Stress is associated with the strength of the body's substance because stress is an indicator of deformation of the body (Hibbeler, 2010).

Everything within the volume of the body at all points. The components of stress differ over the volume of the plate, and they must be in equilibrium with the external forces on the boundary of the plate when we arrive at the boundary, so that external forces can be seen as a continuation of the distribution of internal stress (Timoshenko et al., 1951).

The small hypothesis of displacement handles one the essential fundamentals of solid mechanics, called the superposition principle. Whenever the quantity (stress or displacement) to be calculated is a linear function of the loads that generate it, this concept is true. It is difficult to obtain such accurate solution. It was also observed that the results of the exact theory do not vary significantly for a slender beam from that of material mechanics or elementary approach, given the solutions close to the ends are not needed (Armenakas, 2016).

Plane stress called two dimensional in this case, There is stress only on the plane faces of the component and all stress and forces is parallel to the surface axis. or a thin plate forces effect on boundary parallel to the plan out of the plane stress are zero on both face (Armenakas, 2016). The evaluation of the stress in the slab in plane has also been performed in conjunction with a realistic project through several programs like SAP2000 and SATWE (Hu et al., 2012). The simplest triangular plane stress element (CST) is the constant strain triangle. The constant strain triangle is widely used for various analysis purposes (Kansara, 2004). The quadratic quadrilateral element can be used for the stress of planes because it has both local and global coordinates as a two-dimensional finite element. The roles are quadratic in any direction of the plane (Kattan, 2007).

The most efficient solution is the expansion of the Taylor series or the expansion of the binomial series (Gao et al., 2015). In order to assess the accuracy and efficacy of the proposed sensitivity reanalysis process, new sensitivity reanalysis of static displacement and Taylor series expansion, including size sensitivity and shape sensitivity, will be implemented. For small modifications and large modifications, reanalysis error is less than 1% and 3%, respectively, when compared with The Kirsch and Papalambros method (Zuo et al., 2016).

Shear influences on beam deflections have been found to be only applicable for some very small beams of less than 10 in length to segment dimension. In general, then, material theory mechanics neglects the effects of shear force when measuring beam deflections. However, in terms of beam stresses, the inner shear strength must create the resulting skin stress distribution around the cross – sections (Sadd, 2009). The magnitude of the increased shear stress is associated to the intensity of the bending moment in the chosen component and the rate of transverse change (Zhou et al., 2020).

In the rigid body mechanics where point forces are used to substitute a distribution of force, we have the analogous elasticity situation where, as a consequence of the theory of St. Venant, a distribution of surface traction over a comparatively small part of a boundary can be replaced by a statically equivalent method without altering the distribution of stress at points far enough away from the boundary (Dym et al., 1973). Both beam theories have the same overestimate of the central deflection relative to the theoretical elasticity approach for the uniformly loaded and simply supported case (Levinson, 1981). The non-local Timoshenko beam solutions presented should be beneficial to engineers who design mechanical micro- and nano-electrical devices (Wang et al., 2008). The specifications for the shear rigidity of the models were defined as the span length, panel depth and rotating cross section shear module

(Rahman et al., 2020). The Timoshenko beam section with increased strains will easily capture reinforced concrete columns with less elements, with flexure and shear results (Feng and Wu, 2020).

3. Materials and Methods

The thin plate of plane stress is filled with forces on the border, parallel to the plate plane and evenly distributed over the thickness. The σ_z, τ_{xz}, and τ_{yz} stress components are zero on the both faces, the state stress are σ_x, σ_y and τ_{xy} as shown in figure below (Timoshenko et al., 1951).

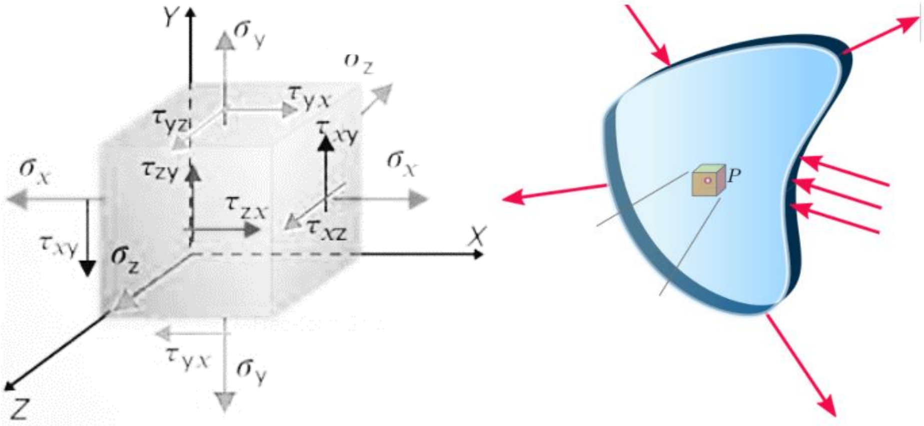


Figure 1 stress components at a point

Relationships to solve plane stress problems.

1. Strain displacement relation:

The displacement field at all nodes inside the solid elements is defined by the continuum hypothesis. Specific measurements of deformation can be built using suitable geometry, leading to the development of the strain tensor. The strain components are as predicted, related to the displacement field. When the relative displacements between points in the body are shifted, an elastic solid is said to be deformed or strained. In comparison with rigid-body motion, this is where the distance between points remains the same (Sadd, 2009).

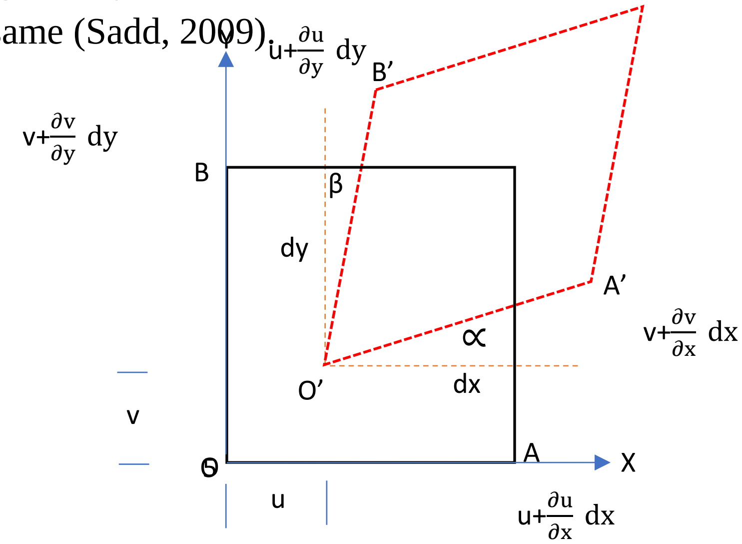


Figure 2 Two-dimensional geometric strain deformation.

Taylor series expansion around point O' to express the components of u and v as:

$$u = u + \frac{\partial u}{\partial x}\, dx, \qquad v = v + \frac{\partial v}{\partial y}\, dy \qquad \text{(Sadd, 2009).}$$

$$\epsilon_x = \frac{O'A' - OA}{OA}, \quad O'A' = dx - u + u + \frac{\partial u}{\partial x}\, dx$$

$$\epsilon_x = \frac{dx - u + u + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\partial u}{\partial x} \quad \therefore \quad \epsilon_x = \frac{\partial u}{\partial x} \qquad \text{.......... (1.1)}$$

$$\epsilon_y = \frac{O'B' - OB}{OB}, \quad O'B' = dy - v + v + \frac{\partial v}{\partial y}\, dy$$

$$\epsilon_y = \frac{dy - v + v + \frac{\partial v}{\partial y} dy}{dy} = \frac{\partial v}{\partial y} \quad \therefore \quad \epsilon_y = \frac{\partial v}{\partial y} \qquad \text{......... (1.2)}$$

$$\gamma_{xy} = \beta + \propto, \qquad Tan\, \beta = \beta, \quad Tan \propto = \propto$$

$$\propto = \frac{v + \frac{\partial v}{\partial x} dx - v}{dx} = \frac{\partial v}{\partial x}, \qquad \beta = \frac{u + \frac{\partial u}{\partial x} dy - u}{dy} = \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \text{...................... (1.3)}$$

ϵ_x = strain x ϵ_y = strain y γ_{xy} = shear strain plane x direction y

2. Plane stress equation (stress strain relation):

σ_x = stress X Direction, σ_y = stress Y Direction, γ_{xy} = shear strain plane x direction y σ_x, σ_y, γ_{xy} are exit in plane stress E= modulus of elasticity G = shear modulus τ = shear stress μ = Poisson ratio. They are defined in section

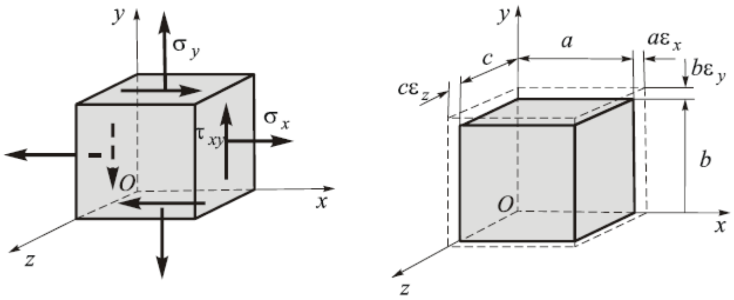


Figure 3 Element of material in plane stress and normal strains

(Examples And Problems In Mechanics Of Materials Stress-Strain State At A Point Of Elastic Deformable Solid Editor-In-Chief Yakiv Karpov, 2010)

$\epsilon_x = \frac{\sigma_x}{E}$ Hooke's Law $\epsilon_y = \frac{-\mu\sigma_x}{E}$

$\gamma_{xy} = \frac{\tau_{yx}}{G}$ $\gamma_{xy} = \frac{E}{2(1+\mu)}$ (Cowin, 2001)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \mu\,\sigma_y) \rightarrow E\,\epsilon_x = (\sigma_x - \mu\,\sigma_y) \quad \ldots\ldots\ldots\ldots\ldots (2.1)$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \mu\,\sigma_x) \rightarrow E\,\epsilon_y = (\sigma_y - \mu\,\sigma_x) \quad \ldots\ldots\ldots\ldots\ldots (2.2)$$

from equation 5 $\rightarrow E\,\epsilon_y + \mu\,\sigma_x = \sigma_y$ .... (2.3)

Add equation (2.3) to equation (2.1) to obtain equation (2.4)

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\,\epsilon_y) \quad \ldots\ldots\ldots\ldots\ldots (2.4)$$

$$\sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\,\epsilon_x) \quad \ldots\ldots\ldots\ldots\ldots (2.5)$$

$$\tau_{xy} = \tau_{yx} = \frac{E}{2(1+\mu)}\gamma_{xy} \quad \ldots\ldots\ldots\ldots\ldots (2.6)$$

Methods for displacement analysis:

A. Theory of elasticity for cantilever end point load.

There are a sample to drive and calculation displacement for any location as shown in below figure (4) by theory of elasticity relations and equations.

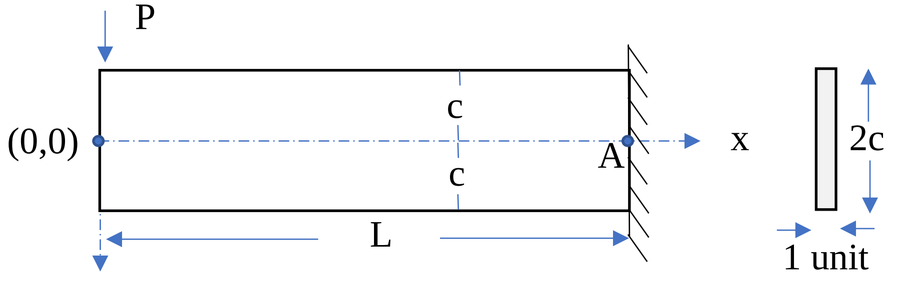


Figure 4 cantilever beam narrow rectangular cross section

From stress strain relationship and Hooke’s Law

$\epsilon_x = \frac{\partial u}{\partial x}$ from equation (1.1) 7and $\epsilon_x = \frac{\sigma_x}{E}$ Hooke's Law

$\epsilon_y = \frac{\partial v}{\partial y}$ from equation (2) and $\epsilon_y = \frac{-\mu\sigma_x}{E}$

Stress function in the form of the polynomial forth degree (Timoshenko and Goodier, 1951) :

$$\emptyset_4 = \frac{a_4}{4*3} x^4 + \frac{b_4}{3*2} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3*2} x y^3 + \frac{e_4}{4*3} y^4$$

Coefficient a_4 …… d_4 in this expression are arbitrary, all of them are zero except d_4 (Timoshenko and Goodier, 1951).

$$\sigma_x = \frac{\partial\emptyset^2}{\partial y^2} = d_4\, xy$$

$$\sigma_y = 0$$

$\tau_{xy} = -\frac{\partial\emptyset^2}{\partial x\,\partial y} = -\frac{d_4}{2} y^2$ by superposing pure the shear $\tau_{xy} = -b_2$

$$\therefore\ \tau_{xy} = -b_2 - \frac{d_4}{2} y^2$$

$\tau_{xy} = 0$ at y= $\pm$c (at boundary condition)

$$0 = -b_2 - \frac{d_4}{2} y^2$$

$$d_4 = -\frac{2b_2}{c^2}$$

Shear force at the end equal to (P)

$$-\int_{-c}^{c} (\tau_{xy})\,dy = -\int_{-c}^{c} \left(-b_2 - \frac{d_4}{2} y^2\right) dy = P$$

$$b_2 = \frac{3\,P}{4\,C}$$

$$\sigma_x = -\frac{3\ P}{2\,C^3}\, xy$$

$$\tau_{xy} = -\frac{3\,P}{4\,C}\left(1 - \frac{y^2}{c^2}\right)$$

$\left(\frac{2}{3\,C^3}\right)$ is the moment of inertia.

$\therefore\ \sigma_x = \frac{-P\,x\,y}{I}$ at (Timoshenko and Goodier, 1951)

$\tau_{xy} = \frac{-P}{2I}(c^2 - y^2)$.

$\therefore\ \frac{\partial u}{\partial x} = \frac{-P\,x\,y}{EI}$ …………. (A)

$\epsilon_y = \frac{\partial v}{\partial y}$ from equation (2) and $\epsilon_y = \frac{-\mu\sigma_x}{E}$

$\therefore\ \frac{\partial v}{\partial y} = \frac{\mu\,P\,x\,y}{EI}$ …………. (B)

$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ from equation (1.3) and $\gamma_{xy} = \frac{\tau_{xy}}{G}$ Hooke's Law

$\gamma_{xy} = \frac{-P}{2GI}(c^2 - y^2)$

$\therefore\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{-P}{2GI}(c^2 - y^2)$ …………. (C)

By integration of equation (A) and (B), that does not presented any difficulty(Timoshenko and Goodier, 1951).u and v are horizontal and vertical displacements.

$u = \int \frac{-P\,x\,y}{EI}\,dx = \frac{-P\,x^2\,y}{2\,EI} + f(y)$…………. (D) f(y) integration constant to (y)

$v = \int \frac{\mu\,P\,x\,y}{EI}\,dy = \frac{\mu\,P\,x\,y^2}{2EI} + f_1(x)$…………. (E) f(x) integration constant to (x)

$\frac{\partial u}{\partial y} = \frac{-P\,x^2}{2\,EI} + \frac{df(y)}{dy}$ …………. (F) From equation (D)

$\frac{\partial v}{\partial x} = \frac{\mu\,P\,y^2}{2EI} + \frac{df_1(x)}{dx}$ …………. (G) From equation (E)

Equation (F) and (G) add to equation (C)

$[\frac{-P\,x^2}{2\,EI} + \frac{df(y)}{dy}] + [\frac{\mu\,P\,y^2}{2EI} + \frac{df_1(x)}{dx}] = \frac{-P}{2GI}(c^2 - y^2)$

$\rightarrow\ [\frac{-P\,x^2}{2\,EI} + \frac{df_1(x)}{dx}] + [\frac{\mu\,P\,y^2}{2EI} + \frac{Py^2}{2GI} + \frac{df(y)}{dy}] = \frac{-P\,c^2}{2GI}$ …………. (H)

Lets:

$F(x) = [\frac{-P\,x^2}{2\,EI} + \frac{df_1(x)}{dx}]$ …………. (I)

$G(y) = [\frac{\mu\,P\,y^2}{2EI} - \frac{Py^2}{2GI} + \frac{df(y)}{dy}]$ …………. (J)

$K = \frac{-P\,c^2}{2GI}$

$\therefore$ F(x) + G(y) = K   F(x) and G(y) are constants such as (d) and (e), they would vary with x and y respectively.

d + e = K   ………… (K)

from equation (I)

$[\frac{-P\,x^2}{2\,EI} + \frac{df(x)}{dx}] = d \;\rightarrow\; \frac{df_1(x)}{dx} = d + \frac{P\,x^2}{2\,EI}$

$f_1(x) = \frac{P\,x^3}{6\,EI} + d\,.\,x + h$ ………… (L)

from equation (J)

$\frac{\mu P\, y^2}{2EI} - \frac{Py^2}{2GI} + \frac{df(y)}{dy} = e \;\rightarrow\; \frac{df(y)}{dy} = \frac{Py^2}{2GI} - \frac{\mu P\, y^2}{2EI} + e$

$f(y) = \frac{Py^3}{6GI} - \frac{\mu P\, y^3}{6EI} + e.\,y + g$ ………… (M)

By substituting equation (L) and (M) in equation (D) and (E).

$u = \frac{-P\,x^2\,y}{2\,EI} + \frac{Py^3}{6GI} - \frac{\mu P\, y^3}{6EI} + e.\,y + g$ ………… (N)

$v = \frac{\mu P\,x\,y^2}{2EI} + \frac{P\,x^3}{6\,EI} + d.\,x + h$ ………… (O)

u = horizontal displacement and v = vertical displacement

The four constants (d, e, g, and h) may be determining in equation (K) and additional three equations from boundary of the beam. And the three conditions of constraint which are necessary to prevent the beam from moving as a rigid body in the xy-plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero, y = 0 (Timoshenko and Goodier, 1951).

At x = L and y = 0 (at center line) → u = 0 and v = 0,

And there are two options, first is $\frac{\partial v}{\partial x} = 0$ and second is $\frac{\partial u}{\partial y} = 0$ used first option

From equation (N) → g = 0

From equation (O) → $0 = 0 + \frac{P\,L^3}{6\,EI} + d.\,L + h \;\rightarrow\; h = \frac{-P\,L^3}{6\,EI} - d.\,L$ ……. (P)

$v = \frac{\mu P\,x\,y^2}{2EI} + \frac{P\,x^3}{6\,EI} + d.\,x + [\frac{-P\,L^3}{6\,EI} - d.\,L]$

$v = \frac{\mu P\,x\,y^2}{2EI} + \frac{P\,x^3}{6\,EI} - \frac{P\,L^3}{6\,EI} - d\,(L-x)$ ………… (Q)

$\frac{\partial v}{\partial x} = \frac{\mu P\, y^2}{2EI} + \frac{P\,x^2}{2\,EI} + d$ (derivation for equation (Q))

From boundary condition → $0 = 0 + \frac{P\,L^2}{2\,EI} + d$ → $d = -\frac{P\,L^2}{2\,EI}$

From equation (P) → $h = \frac{-P\,L^3}{6\,EI} - [-\frac{P\,L^2}{2\,EI}].\,L$ → $h = \frac{P\,L^3}{3\,EI}$

From equation (K) → $-\frac{P\,L^2}{2\,EI} + e = \frac{-P\,c^2}{2GI}$ → $e = \frac{P\,L^2}{2\,EI} - \frac{P\,c^2}{2GI}$

By substituting (g, d, h, and e) in equation (N) and (O), we can determine displacement at any location on the bean for first option that $\frac{\partial v}{\partial x} = 0$.

$$\left\{\begin{array}{l} \mathbf{u = \frac{-P\,x^2\,y}{2\,EI} + \frac{Py^3}{6GI} - \frac{\mu\,P\,\;y^3}{6EI} + [\frac{P\,L^2}{2\,EI} - \frac{P\,c^2}{2GI}]y} \quad \ldots\ldots\ldots\ldots\ \mathbf{(R)} \\ \mathbf{v = \frac{\mu\,P\,x\,y^2}{2EI} + \frac{P\,x^3}{6\,EI} - \frac{P\,L^2 x}{2\,EI} + \frac{P\,L^3}{3\,EI}} \quad \ldots\ldots\ldots\ldots\ \mathbf{(S)} \end{array}\right\} \text{ First option } \frac{\partial v}{\partial x} = 0$$

for second option:

$x = L$ and $y = 0$ (at center line) → $u = 0$ and $v = 0$ and $\frac{\partial u}{\partial y} = 0$

At from equation (N) → $g = 0$

$$u = \frac{-P\,x^2\,y}{2\,EI} + \frac{Py^3}{6GI} - \frac{\mu\,P\;\,y^3}{6EI} + e.\,y + g \quad \ldots\ldots\ldots\ldots \text{(from (N) equation)}$$

$$\frac{\partial u}{\partial y} = \frac{-P\,x^2}{2\,EI} + \frac{Py^2}{2GI} - \frac{\mu\,P\;\,y^2}{2EI} + e$$

$$0 = \frac{-P\,L^2}{2\,EI} + 0 - 0 + e \qquad \rightarrow e = \frac{P\,L^2}{2\,EI}$$

From equation (K) → $d + \frac{P\,L^2}{2\,EI} = \frac{-P\,c^2}{2GI}$ → $d = \frac{-P\,L^2}{2\,EI} - \frac{P\,c^2}{2GI}$

From equation (O) → $0 = 0 + \frac{P\,L^3}{6\,EI} + [\frac{-P\,L^2}{2\,EI} - \frac{P\,c^2}{2GI}]L + h$ → $h = \frac{P\,L^3}{3\,EI} + \frac{P\,c^2 L}{2GI}$

By substituting (g, d, h, and e) in equation (N) and (O), we can determine displacement at any location on the beam for second option that $\frac{\partial u}{\partial y} = 0$.

$$\left\{ \qquad \right\} \text{ Second option } \frac{\partial u}{\partial y} = 0$$

$$\mathbf{u} = \frac{-P\,x^2\,y}{2\,EI} + \frac{Py^3}{6GI} - \frac{\mu\,P\,y^3}{6EI} + [\frac{P\,L^2}{2\,EI}]y \qquad \ldots\ldots\ldots\ldots\ (T)$$

$$\mathbf{v} = \frac{\mu\,P\,x\,y^2}{2EI} + \frac{P\,x^3}{6\,EI} - \frac{P\,L^2 x}{2\,EI} + \frac{P\,L^3}{3\,EI} + \frac{Pc^2}{2GI}(L\text{-}x) \qquad \ldots\ldots\ (U)$$

B. Calculation for Rotation:

In the first option:

From equation (R) $u = \frac{-\mu P\,y^3}{6\,EI} + \frac{P\,y^3}{6IG} - \frac{P\,c^2}{2GI}y$ at $x = L$

$$\frac{\partial u}{\partial y} = \frac{-\mu P\,y^2}{2EI} + \frac{P\,y^2}{2IG} - \frac{P\,c^2}{2GI}$$

at (y) = 0 and (x) = L ∴ rotation = $-\frac{P\,c^2}{2GI}$ ……. (V)

In the Second option: similarly, as procedure above.

at (y) = 0 and (x) = L ∴ rotation = 0

C. Theory of elasticity for cantilever free end surface has shear stress.

Usage of the stress function for deflection calculation. A biharmonic partial differential equation is this equation. The biharmonic operator is called the operator ∇^4 and is in Cartesian coordinates. The method is based on the inverse solution concept where we assume a form of the solution to the biharmonic equation. polynomial method called inverse method.

$$\nabla^4 = \nabla^2\,\nabla^2 = \frac{\partial \emptyset^4}{\partial X^4} + \frac{\partial \emptyset^4}{\partial x^2\,\partial y^2} + \frac{\partial \emptyset^4}{\partial y^4}$$

For the Cantilever beam (figure 4) assume a state of plane stress. The beam is loaded by shear stresses (P) on the free end surface, the stress function is:

$\emptyset = Axy + Bxy^3$ (it is stress function) (Timoshenko and Goodier, 1951).

$$\sigma_x = \frac{\partial \emptyset^2}{\partial y^2} = 6Bxy\,, \quad \sigma_y = \frac{\partial \emptyset^2}{\partial x^2} = 0\,, \quad \tau_{xy} = -\frac{\partial \emptyset^2}{\partial x\,\partial y} = -A-3By^2$$

The constants (A) and (B) will now be determined from the boundary conditions:

$\sigma_y = 0$ $\quad \tau_{xy} = 0$ at $y = \pm c$ and $\quad \tau_{xy} = -p$

$0 = -A-3By^2 \rightarrow A = -3Bc^2$ ………..(1)

$\int_{-c}^{c}(-A-3By^2)dy = -P \rightarrow -2Ac-3B[\frac{y^3}{3}]_{-c}^{c} = -P \rightarrow -2Ac - 2Bc^3 = -P$……..(2)

$B = \frac{-P}{4c^3}$, $A = \frac{3P}{4c}$ from equation (1) and (2).

$\sigma_x = \frac{-3P}{2c^3}xy$, $\sigma_y = 0$, $\tau_{xy} = -\frac{P}{4c}(3-\frac{y^2}{c^2})$

$\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_x = \frac{\sigma_x}{E}$ $\rightarrow \therefore \frac{\partial u}{\partial x} = \frac{-3P}{2Ec^3}xy$ ……… (3)

$\epsilon_y = \frac{\partial v}{\partial y}$, $\epsilon_y = \frac{-\mu\sigma_x}{E}$ $\rightarrow \therefore \frac{\partial v}{\partial y} = \frac{3\mu P}{2Ec^3}xy$ ……….(4)

$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ and $\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)\tau_{xy}}{E}$ Hooke's Law

$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{-(1+\mu)P}{2cE}(3-\frac{y^2}{c^2})$ …….(5)

By integration of equation (3) and (4)

$u = \int \frac{-3P}{2Ec^3}xy\,dx = \frac{-3P}{4Ec^3}x^2y + f(y)$……(6) f(y) integration constant to (y)

$v = \int \frac{3\mu P}{2Ec^3}xy\,dy = \frac{3\mu P}{4Ec^3}xy^2 + f(x)$…(7) f(x) integration constant to (x)

$\frac{\partial u}{\partial y} = \frac{-3P}{4Ec^3}x^2 + \frac{df(y)}{dy}$ …………(8) From equation (6)

$\frac{\partial v}{\partial x} = \frac{3\mu P}{4Ec^3}y^2 + \frac{df(x)}{dx}$ …………(9) From equation (7)

Equation (8) and (9) add to equation (5)

$\frac{-3P}{4Ec^3}x^2 + \frac{df(x)}{dx} + \frac{3\mu P}{4Ec^3}y^2 + \frac{df(y)}{dy} = \frac{-(1+\mu)P}{2cE}(3-\frac{y^2}{c^2})$

$\rightarrow [\frac{-3P}{4Ec^3}x^2 + \frac{df(x)}{dx}] + [\frac{3\mu P}{4Ec^3}y^2 - \frac{(1+\mu)Py^2}{2Ec^3} + \frac{df(y)}{dy}] = \frac{-3(1+\mu)P}{2cE}$

Lets:

$$F(x) = [\frac{-3P}{4Ec^3}x^2 + \frac{df(x)}{dx}] \qquad \text{............} (10)$$

$$G(y) = [\frac{3\mu P}{4\,Ec^3}y^2 - \frac{(1+\mu)Py^2}{2Ec^3} + \frac{df(y)}{dy}] \qquad \text{............} (11)$$

$$K = \frac{-3(1+\mu)P}{2cE}$$

$\therefore F(x) + G(y) = K$ F(x) and G(y) are constants such as (d) and (e), they would vary with x and y respectively.

$$d + e = K \qquad \text{............} (12)$$

from equation (10)

$$[\frac{-3P}{4Ec^3}x^2 + \frac{df(x)}{dx}] = d \;\rightarrow\; \frac{df(x)}{dx} = d + \frac{3P}{4Ec^3}x^2$$

$$f(x) = \frac{P}{4Ec^3}x^3 + d.\,x + h \qquad \text{............} (13)$$

from equation (11)

$$[\frac{3\mu P}{4\,Ec^3}y^2 - \frac{(1+\mu)Py^2}{2Ec^3} + \frac{df(y)}{dy}] = e \;\rightarrow\; \frac{df(y)}{dy} = \frac{-3\mu P}{4\,Ec^3}y^2 + \frac{(1+\mu)Py^2}{2Ec^3} + e$$

$$f(y) = \frac{-\mu Py^3}{4\,Ec^3} + \frac{(1+\mu)Py^3}{6Ec^3} + e.\,y + g \qquad \text{............} (14)$$

By substituting equation (13) and (14) in equation (6) and (7).

$$u = \frac{-3P}{4Ec^3}x^2y - \frac{\mu Py^3}{4\,Ec^3} + \frac{(1+\mu)Py^3}{6Ec^3} + e.\,y + g \qquad \text{............} (15)$$

$$v = \frac{3\mu P}{4\,Ec^3}xy^2 + \frac{P}{4Ec^3}x^3 + d.\,x + h \qquad \text{............} (16)$$

The four constants (d, e, g, and h) may be determining in equation (12) and additional three equations from boundary of the beam. And the three conditions of constraint which are necessary to prevent the beam from moving as a rigid body in the xy-plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero, y = 0 (Timoshenko et al., 1951).

At x = L and y = 0 (at center line) → u = 0 and v = 0, and $\frac{\partial v}{\partial x} = 0$

From equation (15) → g = 0

From equation (16) $\rightarrow$ $0 = 0 + \frac{P}{4Ec^3}L^3 + d.L + h \rightarrow h = \frac{-P}{4Ec^3}L^3 - d.L$ ..(17)

$$v = \frac{3\mu P}{4\,Ec^3}xy^2 + \frac{P}{4Ec^3}x^3 - \frac{P}{4Ec^3}L^3 - d(L-x)$$

$$v = \frac{3\mu P}{4\,Ec^3}xy^2 + \frac{P}{4Ec^3}x^3 - \frac{P}{4Ec^3}L^3 - d\,(L-x) \qquad \text{............(18)}$$

$\frac{\partial v}{\partial x} = \frac{3\mu P}{4\,Ec^3}y^2 + \frac{3P}{4Ec^3}x^2 + d$ (derivation for equation (18))

From boundary condition $\rightarrow$ $0 = 0 + \frac{3P}{4Ec^3}L^2 + d \rightarrow d = \frac{-3P}{4Ec^3}L^2$ From equation (17) $\rightarrow h = \frac{-P}{4Ec^3}L^3 - [\frac{-3P}{4Ec^3}L^2].L \rightarrow h = \frac{P}{2Ec^3}L^3$

From equation (12) $\rightarrow \frac{-3P}{4Ec^3}L^2 + e = \frac{-3(1+\mu)P}{2cE} \rightarrow e = \frac{3P}{4Ec^3}L^2 - \frac{3(1+\mu)P}{2cE}$

By substituting (g, d, h, and e) in equation (15) and (16), we can determine displacement at any location on the bean by stress function that $\frac{\partial v}{\partial x} = 0$.

$$\begin{cases} \mathbf{u} = \frac{-3P}{4Ec^3}x^2y - \frac{\mu P y^3}{4\,Ec^3} + \frac{(1+\mu)Py^3}{6Ec^3} + [\frac{3P}{4Ec^3}L^2 - \frac{3(1+\mu)P}{2cE}]\,y & \text{............(19)} \\ \mathbf{v} = \frac{3\mu P}{4\,Ec^3}xy^2 + \frac{P}{4Ec^3}x^3 - [\frac{3P}{4Ec^3}L^2]\,x + \frac{P}{2Ec^3}L^3 & \text{............(20)} \end{cases}$$

D. Virtual-Work method analysis:

This method used as elementary mechanic for displacement calculation, displacement for end beam in figure (4) determined as bellow, capital (M) illustrate moment from external loads and small (m) from moment by one-unit load and moment for rotation:

$$v = \frac{1}{EI}\int_0^x M.m\,dx \rightarrow v = \frac{1}{EI}\int_0^x (-Px).(-x)dx \rightarrow v = \frac{1}{EI}\left[\frac{px^3}{3}\right]_0^x$$

$$\theta = \frac{1}{EI}\int_0^x M.m\,dx \rightarrow \theta = \frac{1}{EI}\int_0^x (-Px).(-1)dx \rightarrow \theta = \frac{1}{EI}\left[\frac{px^2}{2}\right]_0^x$$

$$\therefore v = \frac{PL^3}{3EI} \quad \text{vertical displacement} \quad \text{if } x = L, \text{it is end of beam.}$$

$$\therefore \theta = \frac{PL^2}{2EI} \quad \text{rotation} \quad \text{if } x = L, \text{it is end of beam.}$$

E. FEM and other methods used to displacement analysis:

In this review literature other methods illustrated in the section result and discissions, that, they are used to calculation displacement. All of them was used to comparative with theory of elasticity displacement calculation results. Such as, the method of stiffness can be used to analyze both statically determined and indeterminate structures, directly producing displacements and forces (Hibbeler, 2012). The computer programs powerful to add shear area effect on stiffness method such as sap2000 program (Sap, 2004).

And For modeling axisymmetric structures under axisymmetric loading, the solid element is a three or four-node element. It is based on an isoperimetric formulation that includes four compatible bending modes that are optional. The solid element a model the mid-plane of an axisymmetric structure's representative sector whose stresses and strains do not differ in the circumferential direction. For the material properties of each solid, orthotropic properties are used(Sap, 2004).

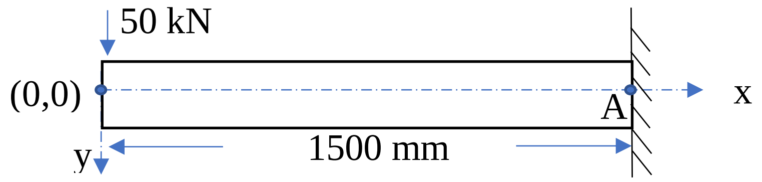
The method of constant strain triangle (CST) is the simplest triangular plane stress element. At each node, this element has two degrees of freedom in the plane, for a total of six degrees of freedom per element. For different purposes of research, the constant strain triangle is commonly used (Kansara, 2004). and the last method is FEM (Finite element method). all of them used by computer.

Using the Autodesk Inventor for FEM analysis. Autodesk Inventor is a platform for 3D mechanical solid modeling design created in 1999 by Autodesk to build interactive 3D prototypes. It used for mechanical modeling, product simulation, and the development of tools. Even before the goods are developed, it will assist you greatly in simulation and visualization. Inventor is a CAD application powered by measurements and is used in engineering projects, modeling of visualization, and documentation. Inventor is a wonderful choice in digital space for representing objects.

4.Results and Discussions

The displacement at the loaded end (x = 0) the value from equation (S) first option and equation (20), this ($PL^3/3EI$) is remained only coincides with the value usually derived in elementary books on the strength of materials, that is verified with it. but in equation (U) second option the deflection is more by effect of shear on beam deflection. The Poisson ratio not effect in all options at center line but it sensitive from another location on beam. The shear modulus not effect at any location on beam in first option for vertical displacement, but it has effect in all options for vertical and horizontal displacement in out of center line.

The case study sample is the steel cantilever beam, that modulus of elasticity $21*10^4$ N/mm^2, Poisson ratio is (0.3), and the dimension sample are (1500*400*10) mm length, height and width respectively as shown below figure.

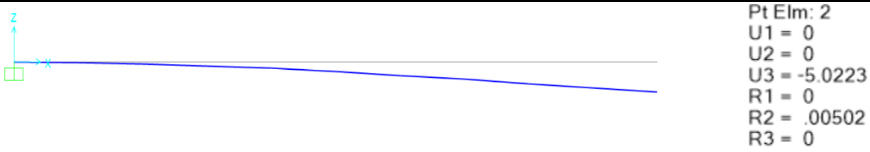


In the table (1) the all options in vertical displacement by theory of elasticity and polynomial stress function method at the center line are the same with Elementary book in mechanic and SAP2000 program (stiffness method), but in models contain effect of shear displacement was more difference with first option, as well as near for second option. but other methods model as (plane stress, solid element and FEM) have a little different with other methods and options, because they are more accurate methods, 3D models, meshing element and support condition and they are not result for rotation. the result of programs was illustrated in figure (5).

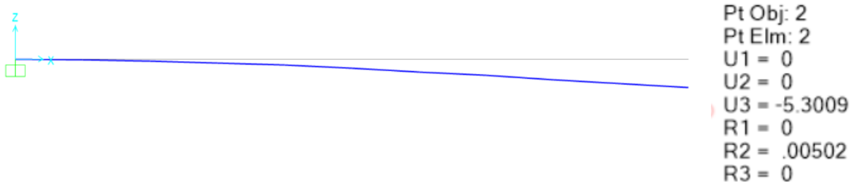
Table 1 The displacement and rotation results and the difference between methods.

No	The Methods of analysis	v (vertical displacement) mm	Difference % with first option	Rotation (rad)
1	First option derived in equations (S), (V) and (20)	5.022	…..	0.0023

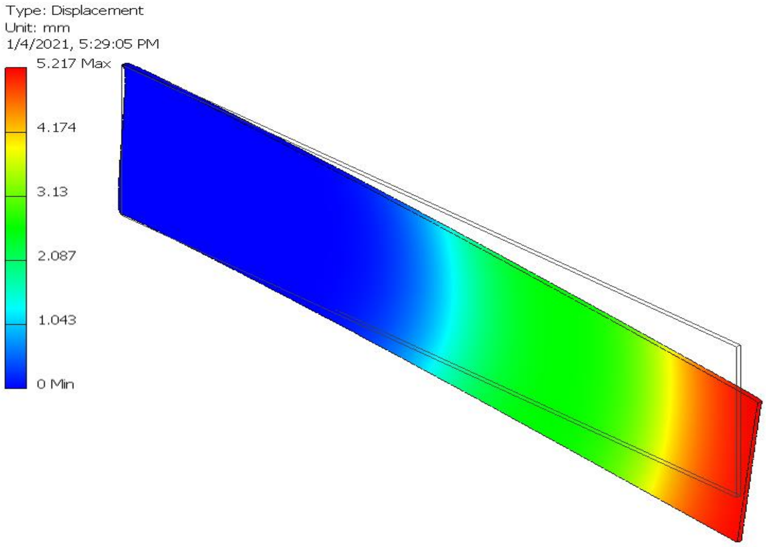
2	Second option derived in equations (U)	5.37	6.93	0.0
3	Elementary book in mechanic (virtual work)	5.022	0.0	0.00502
4	SAP2000 as beam (stiffness method)	5.022	0.0	0.00502
5	Sap 2000 model as plane stress	5.134	2.23	…..
6	SAP2000 model as beam contain effect shear	5.3	5.54	0.00502
7	Sap 2000 model as solid element	5.045	0.46	…..
8	Autodesk inventor program for FEM	5.217	3.88	….



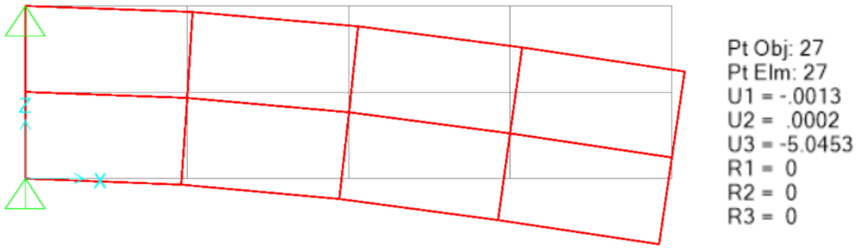
(1) The model of Sap2000 for case study displacement and rotation results.



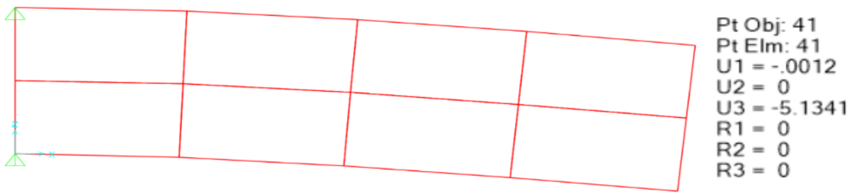
(2) The Shear effect of The Sap2000 model for case study displacement and rotation results.



(3) The FEM of The Autodesk inventor program model for case study displacement results.



(4) The solid model sap2000 program for case study displacement results.



(5) The plane stress model sap2000 program for case study displacement results.

Figure 5 Computer Result Displacements: 1 stiffness method only,2 stiffness method contains shear effect, 3 FEM by Autodesk inventor, 4 solid method and 5 plane stress method.

5.Conclusion

By literature reviews and experimental study, some conclusions were achieved:

1. The first option outlined in the methodology section and polynomial method, as satisfied with the mechanical equations. Shera can affect on beam deflection, as well as Shear stress cause to distortion of the cross section in the end.
2. Theoretical mechanics neglects the effects of shear force when calculating beam deflections (displacement). But the theory of elasticity measures the effect of shear on deflection, which is significant for beams that are very short. But the exact principle does not vary significantly for a slender beam from that of material mechanics or elementary technique.
3. Displacement by elasticity theory is the same with the elementary book of mechanics and stiffness method at the center line of element calculation. But the principle of elasticity for all beam positions is more accurate and sensitive.
4. The Rotation in the end beam by theory of elasticity is lower than the elementary book methods.
5. In the SAP2000 model as beam contain effect shear do not have effect on rotation as well as it has effect on displacement. Solid and plane stress in sap2000 model do have rotation.
6. There is no difficulty in integrating the equation for each individual strain to displacement relationship, since more constants in linear equation components mean that displacement does not only determine stresses and strain.
7. The Poisson ratio does not affect the center line beam in all options, but is sensitive to the beam from another location.
8. In the elasticity theory of measurement displacement, the equations of the second option, all conditions are fulfilled with only some distortion problem at the fixed end beam and it has some terms cause to is not equal to the geometrical solutions on the strength of materials.

9. The end cantilever distribution forces are not the right response, but can be considered appropriate for cross sections at a typically avoid from the ends, according to the Saint-Venant theory.
10. The results solid method was near than the results of displacement elasticity equations theory. In the sap2000 model containing the shear effect, more displacements are caused as compared to material strength in elementary books and the theory of elasticity first option equations, but same for second option.

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